Section 1-2, Mathematics 108

Integer Exponents

We define an integer exponent n over a base $a \neq 0$ as follows:

$$a^n = \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{n \text{ times}}$$

So if we multiply two exponential expressions with the same base

$$a^n a^m = \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{n \text{ times}} \times \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{m \text{ times}} = \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{n+m \text{ times}} = a^{n+m}$$

(1) $\underline{a^n a^m = a^{n+m}}$

Similarly if $n > m$ we can divide

$$\frac{a^n}{a^m} = \frac{a \cdot a \cdot a \cdots a \cdot a \; n-times}{a \cdot a \cdot a \cdots a \cdot a \; m-times} = \left\{ \frac{a}{a} \cdot \frac{a}{a} \cdot \frac{a}{a} \cdots \frac{a}{a} (m-times) \right\} \cdot \frac{a \cdot a \cdot a \cdots a}{(n-m) times} = a^{n-m}$$

(2) $\underline{\dfrac{a^n}{a^m} = a^{n-m}}$

Note that $\dfrac{a^n}{a^n} = 1$ so applying the previous equation we see that $1 = \dfrac{a^n}{a^n} = a^{n-n} = a^0$

(3) for $\underline{a \neq 0 \quad a^0 = 1}$

Using this preceding

$$\frac{1}{a^n} = \frac{a^0}{a^n} = a^{0-n} = a^{-n}$$

(4) $\underline{\dfrac{1}{a^n} = a^{-n}}$

Other properties of exponents

$$\left(a^n\right)^m = \underbrace{\left(\begin{matrix} a\cdot a\cdot a\cdots a\cdot a \\ n-times \end{matrix}\right)\cdot\left(\begin{matrix} a\cdot a\cdot a\cdots a\cdot a \\ n-times \end{matrix}\right)\cdots\left(\begin{matrix} a\cdot a\cdot a\cdots a\cdot a \\ n-times \end{matrix}\right)}_{m-times} = \begin{matrix} a\cdot a\cdot a\cdots a\cdot a \\ nm-times \end{matrix} = a^{nm}$$

(5) $\underline{\left(a^n\right)^m = a^{nm}}$

$$\left(ab\right)^n = \begin{matrix} ab\cdot ab\cdots ab \\ n-times \end{matrix} = \begin{matrix} a\cdot a\cdots a \\ n-times \end{matrix} \times \begin{matrix} b\cdot b\cdots b \\ n-times \end{matrix} = a^n b^n$$

(6) $\underline{\left(ab\right)^n = a^n b^n}$

$$\left(\frac{a}{b}\right)^n = \frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\cdots\frac{a}{b} = \frac{a\cdot a\cdot a\ldots a}{b\cdot b\cdot b\ldots b} = \frac{a^n}{b^n}$$

(7) $\underline{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$

$$\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}} = \frac{\frac{1}{a^n}}{\frac{1}{b^n}} = \frac{b^n}{a^n} = \left(\frac{b}{a}\right)^n$$

(8) $\underline{\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n}$

$$\frac{a^{-n}}{b^{-m}} = \frac{\frac{1}{a^n}}{\frac{1}{b^m}} = \frac{b^m}{a^n}$$

(9) $\underline{\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}}$

Summary

$a^n a^m = a^{n+m}$	$\dfrac{a^n}{a^m} = a^{n-m}$	for $a \neq 0 \quad a^0 = 1$
$\dfrac{1}{a^n} = a^{-n}$	$\left(a^n\right)^m = a^{nm}$	$\left(ab\right)^n = a^n b^n$
$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$	$\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}$

Example

Simplify $\left(\dfrac{x}{y}\right)^3 \left(\dfrac{y^3 x}{z}\right)^4$

$$\left(\frac{x}{y}\right)^3 \left(\frac{y^3 x}{z}\right)^4 = \frac{x^3}{y^3} \cdot \frac{\left(y^3\right)^4 x^4}{z^4} = \frac{x^3 y^{12} x^4}{y^3 z^4} = \frac{x^{3+4} y^{12-3}}{z^4} = \underline{\frac{x^7 y^9}{z^4}}$$

Fractional exponents

We start with the question, what does $5^{1/2}$ mean?

If $x = 5^{1/2}$ then $x \cdot x = 5^{1/2} \cdot 5^{1/2} = 5^{1/2+1/2} = 5^1 = 5$

So we have $x^2 = 5$ so $x = \sqrt{5}$

Similarly

(1) $\underline{a^{1/n} = \sqrt[n]{a}}$

What then is $a^{n/m}$?

$$a^{n/m} = \left(a^n\right)^{1/m} = \sqrt[m]{a^n}$$

but also

$$a^{n/m} = \left(a^{1/m}\right)^n = \left(\sqrt[m]{a}\right)^n$$

(2) $\underline{a^{n/m} = \sqrt[m]{a^n} = \left(\sqrt[m]{a}\right)^n}$

Other properties of nth roots

$$\sqrt[n]{ab} = \left(ab\right)^{1/n} = a^{1/n}b^{1/n} = \sqrt[n]{a}\sqrt[n]{b}$$

(3) $\underline{\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}}$

$$\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{1/n} = \frac{a^{1/n}}{b^{1/n}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

(4) $\underline{\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}}$

$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[n]{a^{1/m}} = \left(a^{1/m}\right)^{1/n} = a^{1/nm} = \sqrt[nm]{a}$$

(5) $\underline{\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}}$

Lastly, consider $\sqrt[n]{a^n}$ where n is even and $a \in \mathbb{Z}$.

Since n is even, $a^n \geq 0$, so $\sqrt[n]{a^n} \geq 0$ but $\sqrt[n]{a^n} = a^{n/n} = a^1 = a$ so

(6) $\sqrt[n]{a^n} = |a|$ for n even

Summary

$a^{1/n} = \sqrt[n]{a}$	$a^{n/m} = \sqrt[m]{a^n} = \left(\sqrt[m]{a}\right)^n$	$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$
$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$	$\sqrt[n]{a^n} = \|a\|$ for n even